Basic Sine, Cosine, and Tangent Equations

Basic steps for solving cos $x = a$:

1. **Find all the angles on the unit circle (on $[0, 2\pi]$) that satisfy the equation.** One of these solutions will be $s = \cos^{-1} a$ and the other will be $2\pi - s = 2\pi - \cos^{-1} a$.
2. **Add or subtract multiples of 2π from each angle.**

Basic steps for solving sin $x = a$:

1. **Find all the angles on the unit circle (on $[0, 2\pi]$) that satisfy the equation.** You can do this by looking at the unit circle (usually this is less confusing) or by working algebraically. If $s = \sin^{-1} a > 0,$ one of these solutions will be $s = \sin^{-1} a$ and the other will be $\pi - s = \pi - \sin^{-1} a$. If $s = \sin^{-1} a < 0,$ the two solutions are $s + 2\pi = \sin^{-1} a + 2\pi$ and $\pi - s = \pi - \sin^{-1} a.$
2. **Add or subtract multiples of 2π from each angle.**

Don't let the algebra freak you out! All you are doing is finding all the angles on the unit circle that satisfy the equation and adding $2k\pi$ to each one to form your solution set.

Basic steps for solving tan $x = a$:

1. **Find one angle on the unit circle that satisfies the equation.** This will be either $s = \tan^{-1} a$ if this value is positive, or $s + \pi = \tan^{-1} a + \pi$ if $s = \tan^{-1} a$ is negative.
2. **Add or subtract multiples of π from each angle.** (Remember that the tangent repeats every π instead of every 2π like sine and cosine).

Examples: Find all real numbers that satisfy each equation.

a) $\sin x = 1$

b) $\cos x = 0$

c) $\cos x = -1/2$

d) $\sin x = \sqrt{2}/2$

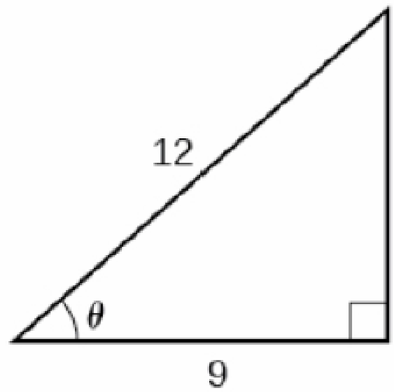
e) $\tan x = -\sqrt{3}$

f) $\tan x = 1$

Examples: Find all angles in $[0°,\ 360°]$ that satisfy the equation

a) $\cos x = \sqrt{3}/2$

Example 6.1.1. Solve the following triangle for the angle θ.



Solution. Because we know the lengths of the hypotenuse and the side adjacent to the angle θ, it makes sense for us to use the cosine function.

$$\cos(\theta) = \frac{9}{12}$$

$$\theta = \cos^{-1}\left(\frac{9}{12}\right) \qquad \text{from properties of the arccosine}$$

$$\theta \approx 0.7227 \text{ radians or } \theta \approx 41.4096^\circ$$

□

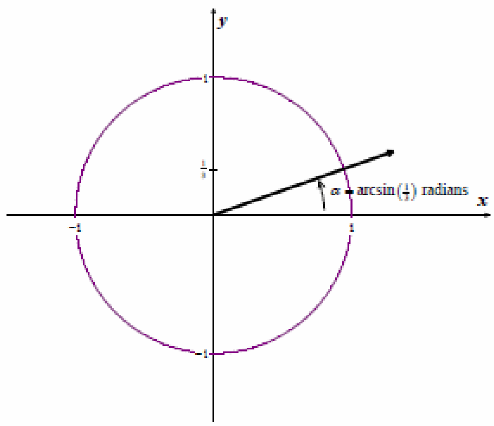
Knowing the measure of one acute angle in a right triangle, we can easily determine the measure of the second acute angle. In the example above, the measure of the angle opposite the side of length 9 would be, approximately, $180^\circ - 90^\circ - 41.4096^\circ = 48.5904^\circ$. Note that the exact measure is $\sin^{-1}\left(\frac{9}{12}\right)$.

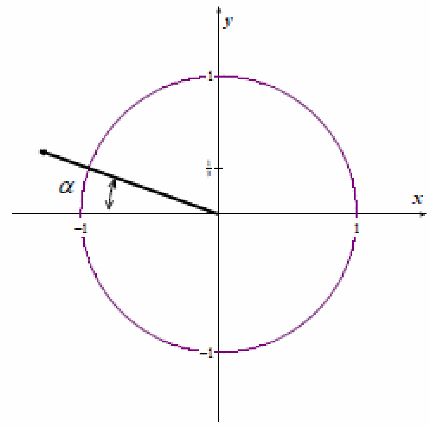
Example 6.1.2. Solve the following equations.

1. Find all angles θ for which $\sin(\theta) = \frac{1}{3}$.

Solution.

1. If $\sin(\theta) = \frac{1}{3}$, then the terminal side of θ, when plotted in standard position, intersects the Unit Circle at $y = \frac{1}{3}$. Geometrically, we see that this happens at two places: in Quadrant I and in Quadrant II. If we let α denote the acute solution to the equation, then all of the solutions to this equation in Quadrant I are coterminal with α, and α serves as the reference angle for all solutions in Quadrant II.





Noting that $\frac{1}{3}$ is not the sine of any of the common angles, we use the arcsine function to express our answers. The real number $t = \arcsin\left(\frac{1}{3}\right)$ is defined so it satisfies $0 < t < \frac{\pi}{2}$ with $\sin(t) = \frac{1}{3}$. Hence, $\alpha = \arcsin\left(\frac{1}{3}\right)$ radians.

Since the solutions in Quadrant I are all coterminal with α, we get part of our solution to be

$$\begin{aligned} \theta &= \alpha + 2\pi k \\ &= \arcsin\left(\frac{1}{3}\right) + 2\pi k, \text{ for integers } k. \end{aligned}$$

Turning our attention to Quadrant II, we get one solution to be $\pi - \alpha$. Hence, the Quadrant II solutions are

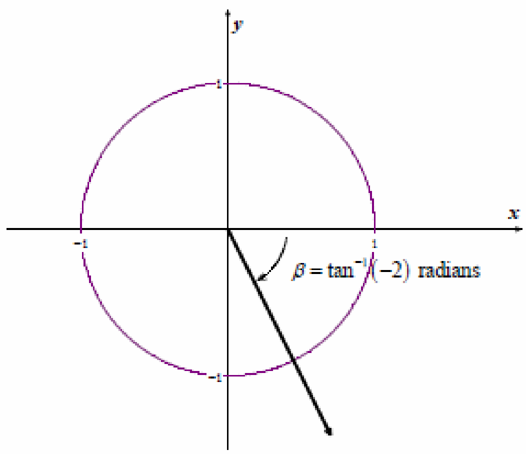
$$\begin{aligned}\theta &= \pi - \alpha + 2\pi k \\ &= \pi - \arcsin\left(\frac{1}{3}\right) + 2\pi k, \text{ for integers } k.\end{aligned}$$

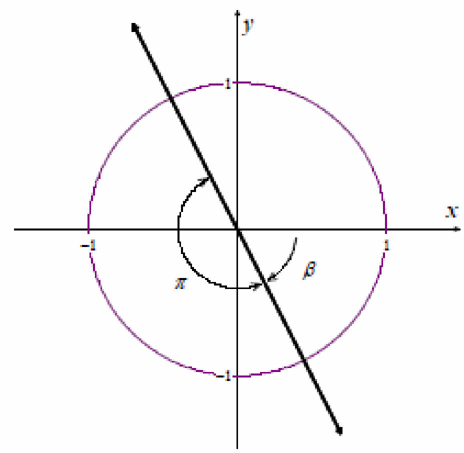
Our final answer is that the solution to $\sin(\theta) = \frac{1}{3}$ is $\theta = \arcsin\left(\frac{1}{3}\right) + 2\pi k$ or $\theta = \pi - \arcsin\left(\frac{1}{3}\right) + 2\pi k$ for all integers k.

2. Find all real numbers t for which $\tan(t) = -2$.

Solution:

2. We may visualize the solutions to $\tan(t) = -2$ as angles θ with $\tan(\theta) = -2$. Since tangent is negative only in Quadrants II and IV, we focus our efforts there.





We note that none of the common angles have tangent –2, so we need to use the arctangent, or inverse tangent, function to express our answers. The real number $t = \tan^{-1}(-2)$ satisfies $\tan(t) = -2$ and $-\frac{\pi}{2} < t < 0$. If we let $\beta = \tan^{-1}(-2)$ radians, we see that all of the Quadrant IV solutions to $\tan(t) = -2$ are coterminal with β. Moreover, the solutions from Quadrant II

differ by exactly π units from the solutions in Quadrant IV, so all the solutions to $\tan(t) = -2$ are of the form

$$\begin{aligned} \theta &= \beta + \pi k \\ &= \tan^{-1}(-2) + \pi k \text{ for some integer } k. \end{aligned}$$

Switching back to the variable *t*, we record our final answer to $\tan(t) = -2$ as $t = \tan^{-1}(-2) + \pi k$ for integers *k*.

3. Solve $\sec(x) = -\frac{5}{3}$ for *x*.

Solution:

3. The last equation we are asked to solve, $\sec(x) = -\frac{5}{3}$, poses an immediate problem. We are not told whether or not *x* represents an angle or a real number. We assume the latter, but note that we will use angles and the Unit Circle to solve the equation regardless.

   Adopting an angle approach, we consider $\sec(\theta) = -\frac{5}{3}$ and note that our solutions lie in Quadrants II and III. Since $-\frac{5}{3}$ isn't the secant of any of the common angles, we'll need to express our solutions in terms of the arcsecant function. The real number $x = \operatorname{arcsec}\left(-\frac{5}{3}\right)$ is defined so that $\frac{\pi}{2} < x < \pi$ with $\sec(x) = -\frac{5}{3}$.

   If we let $\beta = \operatorname{arcsec}\left(-\frac{5}{3}\right)$, we see that β is a Quadrant II angle. To obtain a Quadrant III angle solution, we may simply use $-\beta = -\operatorname{arcsec}\left(-\frac{5}{3}\right)$. Since all angle solutions are coterminal with

β or $-\beta$, we get our solutions for $\sec(\theta) = -\frac{5}{3}$ to be, for integers k,

$$\theta = \beta + 2\pi k \qquad \text{or} \qquad \theta = -\beta + 2\pi k$$

$$\theta = \operatorname{arcsec}\left(-\frac{5}{3}\right) + 2\pi k \qquad \text{or} \qquad \theta = -\operatorname{arcsec}\left(-\frac{5}{3}\right) + 2\pi k.$$

Switching back to the variable x, we record our final answer to $\sec(x) = -\frac{5}{3}$ as

$x = \operatorname{arcsec}\left(-\frac{5}{3}\right) + 2\pi k$ or $x = -\operatorname{arcsec}\left(-\frac{5}{3}\right) + 2\pi k$ for integers k.

Now Try:

1. $\sin x = -.4375$

2. $\cos x = .8913$

3. $\tan x = -3.5$